Prostor Minkovskog

Kvadrivektori

$$(ct, x, y, z) \equiv (ct, \vec{r}) = x^{\mu}$$

$$\left(\frac{E}{c}, p_x, p_y, p_z\right) \equiv \left(\frac{E}{c}, \vec{p}\right) = p^{\mu}$$

kontravarijantni vektori

Metrika prostora

$$ds^2 = g_{\mu\nu}\, dx^{\mu} dx^{\nu}$$

$g_{\mu\nu} \rightarrow$ metrički tenzor

$g_{00} = 1$

$g_{11} = g_{22} = g_{33} = -1 \quad , \quad g_{\mu\nu} = 0, \mu \neq$

$$x_{\mu} = \sum_{\nu=0}^{3} g_{\mu\nu} x^{\nu} \equiv g_{\mu\nu} x^{\nu}$$

Ajnštajnova konvencija

Kovarijantni vektor

$$x_{\mu} = (ct, -\vec{r})$$

$$g_{\mu\nu} g^{\nu\rho} = \delta^{\rho}_{\mu} \quad \text{Kronekerova delta}$$

Lorencove transformacije (grupa)

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} \quad ; \quad \Lambda^{\mu}_{\ \nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$$

$\Lambda^{\mu}_{\ \nu}$ – matrica 4x4

Od interesa su transformacije za koje je ispunjeno da je
$\det \Lambda^{\mu}{}_{\nu} = 1$ i $|\Lambda^{0}{}_{0}| \geq 1$

Onda se Lorencove transformacije mogu zapisati kao

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}} \longrightarrow \text{generatori grupe}$$

$\omega_{\mu\nu}$ ↓ parametri grupe

$\omega_{\mu\nu}$ (6 parametara)
- 3 ugla rotacija
- 3 pravca boostova

Poenkareove transformacije (grupa)

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \longrightarrow \text{translacije}$$

10 parametarska grupa

Dopuna

$\frac{\partial}{\partial ct} = \frac{\partial}{\partial x^0} = \partial_0$ $\frac{\partial}{\partial y} = \frac{\partial}{\partial x^2} = \partial_2$

$\frac{\partial}{\partial x} = \frac{\partial}{\partial x^1} = \partial_1$ $\frac{\partial}{\partial z} = \frac{\partial}{\partial x^3} = \partial_3$

$$\frac{\partial \varphi}{\partial x^\mu} = \partial_\mu \varphi \quad , \quad \frac{\partial \varphi}{\partial x_\mu} = \partial^\mu \varphi$$

$$\partial^\mu = g^{\mu\nu} \partial_\nu$$

$$\partial^\mu \partial_\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box$$

Dalambergian

1. Zapisati eksplicitno matricu za Lorencove transformacije kovarijantnih vektora ako je poznata veza $X_\nu = g_{\mu\nu} X^\mu$ i Lorencove transformacije za kontravarijantne vektore

$$x'^0 = \gamma(x^0 - \beta x^1)$$

$$x'^1 = \gamma(x^1 - \beta x^0) \qquad (*)$$

$$x'^2 = x^2 \ , \quad x'^3 = x^3$$

$$X_\nu = g_{\mu\nu} X^\mu \qquad g_{\mu\nu} = 0 \quad \mu \neq \nu$$

$$(**) \qquad g_{11} = 1 \ , \quad g_{22} = g_{33} = g_{44} = -1$$

$(**) \Rightarrow$

$$X_0 = X^0$$

$$X_1 = -X^1$$

$$X_2 = -X^2$$

$$X_3 = -X^3$$

$(*)$ i $(**) \Rightarrow$

$$X_0' = \gamma(X_0 + \beta X_1)$$

$$-X_1' = \gamma(-X_1 - \beta X_0)$$

$$X_2' = X_2$$

$$X_3' = X_3$$

odnosno

$$x_0' = \gamma (x_0 + \beta x_1)$$

$$x_1' = \gamma (x_1 + \beta x_0) \qquad (***)$$

$$x_2' = x_2 , \qquad x_3' = x_3$$

U matričnom obliku

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} , \quad \beta = \frac{v}{c} , \quad \gamma = \frac{1}{\sqrt{1\ldots}}$$

A što se može dobiti i diferenciranjem ti po pravilu za transformaciju kovarijantnih vektora

$$x'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} x_\nu \qquad (\text{za domaći})$$

Koristeći inverziju (*)

Za domaći

Ako je poznato da se ~~konti~~ kovarijantni vektori transformišu po pravilu

$b'_\mu = L_\mu^{\ \nu} b_\nu$ kako glasi dva puta kontravarijantni tenzor $L^{\nu\mu}$?

$L^{\mu\nu} = g^{\mu\alpha} L_\alpha^{\ \nu}$ (pomnožiti odgovarajuće 4x4 matri

2. Pokazati da se izvodni operator $\frac{\partial}{\partial x^\mu}$ transformiše kao kovarijantni vektor. Koristiti def. za transformacije kontravarijantnih vektora preko Lorencovih transformacija u pravcu $x (\equiv x^1)$ ose.

Def. Lorencovih transformacija (u pravcu $x = x^1$ ose)

$$
\begin{aligned}
x'^0 &= \gamma (x^0 - \beta x^1) \\
x'^1 &= \gamma (x^1 - \beta x^0) \\
x'^2 &= x^2 \\
x'^3 &= x^3
\end{aligned}
\qquad (*)
$$

$$\boxed{x'^\mu = x'^\mu (x^0 \ldots x^3)}$$

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^0}\frac{\partial x^0}{\partial x'^\mu} + \frac{\partial}{\partial x^1}\frac{\partial x^1}{\partial x'^\mu} + \frac{\partial}{\partial x^2}\frac{\partial x^2}{\partial x'^\mu} + \frac{\partial}{\partial x^3}\frac{\partial x^3}{\partial x'^\mu}$$

$$\frac{\partial}{\partial x'^0} = \frac{\partial}{\partial x^0}\frac{\partial x^0}{\partial x'^0} + \frac{\partial}{\partial x^1}\frac{\partial x^1}{\partial x'^0}$$

$$\frac{\partial}{\partial x'^1} = \frac{\partial}{\partial x^0}\frac{\partial x^0}{\partial x'^1} + \frac{\partial}{\partial x^1}\frac{\partial x^1}{\partial x'^1}$$

$$\frac{\partial}{\partial x'^2} = \frac{\partial}{\partial x^2} \qquad \frac{\partial}{\partial x'^3} = \frac{\partial}{\partial x^3}$$

Iz (*)

$$x^0 = \gamma(x'^0 + \beta x'^1) \qquad x^2 = x'^2$$

$$x^1 = \gamma(x'^1 + \beta x'^0) \qquad x^3 = x'^3 \qquad (**)$$

Zbog (**)

$$\frac{\partial}{\partial x'^0} = \gamma \frac{\partial}{\partial x^0} + \gamma\beta \frac{\partial}{\partial x^1} \qquad \frac{\partial}{\partial x'^2} = \frac{\partial}{\partial x^2}$$

$$\frac{\partial}{\partial x'^1} = \gamma\beta \frac{\partial}{\partial x^0} + \gamma \frac{\partial}{\partial x^1} \qquad \frac{\partial}{\partial x'^3} = \frac{\partial}{\partial x^3}$$

U matričnom obliku

$$\begin{pmatrix} \frac{\partial}{\partial x'^0} \\ \frac{\partial}{\partial x'^1} \\ \frac{\partial}{\partial x'^2} \\ \frac{\partial}{\partial x'^3} \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x^0} \\ \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^3} \end{pmatrix}$$

Dakle $\frac{\partial}{\partial x^\mu}$ se transformiše kao kovarijantni vektori i obilježava se kao ∂_μ

3 Koristeći Lorencove transformacije (boostove)

$$x'^0 = \gamma (x^0 - \beta x^1)$$

$$x'^1 = \gamma (x^1 - \beta x^0) \qquad (*)$$

$$x'^2 = x^2 , \quad x'^3 = x^3$$

pokazati da je $ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ Lorencova invarijanta. $\gamma^2 = \frac{1}{1-\beta^2}$

Iz $(*)$

$$dx'^0 = \gamma (dx^0 - \beta dx^1)$$

$$dx'^1 = \gamma (dx^1 - \beta dx^0)$$

$$dx'^2 = dx^2$$

$$dx'^3 = dx^3$$

$$(dx'^0)^2 - (dx'^1)^2 - (dx'^2)^2 - (dx'^3)^2 =$$

$$= \gamma^2 (dx^0 - \beta dx^1)^2 - \gamma^2 (dx^1 - \beta dx^0)^2 - (dx^2)^2 - (dx^3)^2$$

$$= \gamma^2 [(dx^0)^2 - \cancel{2\beta dx^0 dx^1} + \beta^2 (dx^1)^2 - (dx^1)^2 + \cancel{2\beta dx^1 dx^0} - \beta^2 (dx^0)^2] - (dx^2)^2 - (dx^3)^2$$

$$= \gamma^2 [(dx^0)^2 (1-\beta^2) - (1-\beta^2)(dx^1)^2] - (dx^2)^2 - (dx^3)^2$$

$$= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

$ds'^2 = ds^2 \rightarrow$ Lorenc invarijantna veličina

4. Pokazati da Lorencove transformacije zadovoljavaju $\Lambda^T g \Lambda = g$ i da ako se zapišu kao

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

onda su $\omega_{\mu\nu}$ antisimetrični.

Kvadrat dužine četvorovektora je

$$x^2 = g_{\mu\nu} x^\mu x^\nu$$

$$x'^\mu = \Lambda^\mu{}_\rho x^\rho$$

Lorencove transformacije čuvaju vrednost skalara $\boxed{x'^2 = x^2}$, tj.

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma = g_{\rho\sigma} x^\rho x^\sigma$$

odnosno

$$\Lambda^\mu{}_\rho g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma} \qquad (*)$$

ili

$$(\Lambda^T)_\rho{}^\mu g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma} \Longrightarrow$$

$$\Lambda^T g \Lambda = g \qquad (1)$$

Zamenjujući infinitezimalnu formu za Lorencove transformacije $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$

u (*)

$$(\delta^\mu_\rho + \omega^\mu{}_\rho)\, g_{\mu\nu}\, (\delta^\nu_\sigma + \omega^\nu{}_\sigma) = g_{\rho\sigma}$$

$$g_{\rho\sigma} + \omega^\mu{}_\rho\, g_{\mu\nu}\, \delta^\nu_\sigma + \omega^\nu{}_\sigma\, g_{\mu\nu}\, \delta^\mu_\rho + O(\omega^2) = g_{\rho\sigma}$$

$$\Downarrow$$

$$\omega_{\nu\rho}\,\delta^\nu_\sigma + \omega_{\mu\sigma}\,\delta^\mu_\rho = 0$$

odnosno

$$\omega_{\sigma\rho} + \omega_{\rho\sigma} = 0 \Rightarrow \boxed{\omega_{\sigma\rho} = -\omega_{\rho\sigma}}$$

5. Pokazati da je nemoguće za jedan izolovan elektron da apsorbuje ili emituje foton.

Zakon održanja za četvorovektore impulsa fotona (γ) i elektrona (e) (apsorpcija fotona)

$$P_\gamma + P_e = P_e' \qquad (1)$$

Kvadriranjem obje strane (1) ⇒

$$P_\gamma \cdot P_\gamma + 2P_e \cdot P_\gamma + P_e \cdot P_e = P_e' \cdot P_e' \qquad (2)$$

Digresija

$$p_\mu \cdot p^\mu = \left(\frac{E}{c}, -\vec{p}\right)\left(\frac{E}{c}, \vec{p}\right) = \frac{E^2}{c^2} - \vec{p}^{\,2} \qquad (3)$$

Poznato je da važi relacija

$$E^2 = c^2 \vec{p}^{\,2} + m_0^2 c^4$$

odnosno

$$\frac{E^2}{c^2} = \vec{p}^{\,2} + m_0^2 c^2 \Rightarrow \frac{E^2}{c^2} - \vec{p}^{\,2} = m_0^2 c^2$$

pa je (3) $\quad p_\mu \cdot p^\mu = m_0^2 c^2 \qquad (4)$

Zbog (4) iz (2) ⇒

$$0 + 2P_e \cdot P_\gamma + m_e^2 c^2 = m_e^2 c^2 \qquad (5)$$

a iz (5) ⇒

$$P_e \cdot P_\gamma = 0 \qquad (6)$$

Budući da elektron miruje pre apsorpcije važi da je

$$P_e = (m_e c, 0)$$

a za foton važi

$$P_{ge} = \left(\frac{E_{ge}}{c}, \vec{p}\right)$$

Da bi (6) bilo zadovoljeno, a $m_e \neq 0$, mora da bude $E_{ge} = 0$, tj. nema fotona. Dakle, proces apsorpcije se ne može odigrati (Slično i za emitovanje)

6. Pokazati da je zapreminski element

$$\frac{d^3\vec{k}}{\omega_k}$$

Lorentz invarijanta.

$$k^\mu = \left(\frac{\omega_k}{c}, \vec{k}\right)$$ k^0

$$k_0' = \gamma(k_0 + \beta k_1) \qquad (1)$$

$$k_1' = \gamma(k_1 + \beta k_0) \qquad (2)$$

$$k_2' = k_2 \qquad (3)$$

$$k_3' = k_3 \qquad (4)$$

{ kovarijantni vektori

Jakobijan transformacije je

$$J = \frac{\partial(k_1', k_2', k_3')}{\partial(k_1, k_2, k_3)} = \frac{dk_1'}{dk_1}$$

(ovo nije očigledno, Pokazati, za vežbu, po definiciji)

$$J = \left| \;\vdots\; \right| = \gamma$$

$$dk_1' = \sum \frac{\partial f}{\partial k_i} dk_i \quad f\,\text{jed}$$

$$k_1' = f(k_1, k_2, k_3)$$

Iz (2)

$$\frac{dk_1'}{dk_1} = \gamma\left(1 + \beta\frac{dk_0}{dk_1}\right)$$

Veza k_0 i k_1

$$E^2 = \vec{p}^2c^2 + m_0^2c^4$$

$$\frac{E^2}{\hbar^2} = \vec{p}^2\frac{c^2}{\hbar^2} + \frac{m_0^2c^4}{\hbar^2}$$

$$\omega_k^2 = \vec{k}^2 c^2 + \frac{m_0^2 c^4}{\hbar^2}$$

$$k_0^2 = \vec{k}^2 + m_0^2 \quad \boxed{c=1}\ \boxed{\hbar=1} \Rightarrow \quad k_0 = +\sqrt{k_1^2 + k_2^2 + k_3^2 + m_0^2} \quad (5$$

$(5) \Rightarrow$

$$\frac{dk_0}{dk_1} = \frac{1}{2\sqrt{\vec{k}^2 + m_0^2}}\, 2k_1 = \frac{k_1}{\sqrt{\vec{k}^2 + m_0^2}} = \frac{k_1}{k_0}$$

Dakle,

$$\frac{dk_1'}{dk_1} = \gamma\left(1 + \beta\frac{k_1}{k_0}\right) = \frac{\gamma}{k_0}\left(k_0 + \beta k_1\right) \overset{(1)}{=}$$

$$= \frac{\gamma}{k_0}\,\frac{k_0'}{\gamma} = \frac{k_0'}{k_0}$$

Ili

$$\frac{dk_1'}{k_0'} = \frac{dk_1}{k_0} \Rightarrow \boxed{\frac{d^3\vec{k}'}{\omega_{k'}} = \frac{d^3\vec{k}}{\omega_k}}$$

7 Ako je L Lorencov skalar, pokazati da je

$\frac{\partial L}{\partial A_\nu}$ kontravarijantni Lorencov vektor i da je

$\frac{\partial L}{\partial(\partial_\mu A_\nu)}$ kontravarijantni Lorencov tenzor.

Pošto je A_ν kovarijantni vektor, transformiše se kao

$$A'_\nu = \frac{\partial x^\rho}{\partial x'^\nu} A_\rho \quad \Rightarrow$$

$$\cancel{A_\rho = \frac{\partial x'^\mu}{\partial x^\rho} A'_\mu}$$

ЗАНЕМАРИТИ

$$\frac{\partial A_\rho}{\partial A'_\nu} = \cancel{\frac{\partial x'^\mu}{\partial x^\rho} \delta^\nu_\mu} = \frac{\partial x'^\nu}{\partial x^\rho}$$

Zato je

$$\frac{\partial L}{\partial A'_\nu} = \frac{\partial A_\rho}{\partial A'_\nu} \frac{\partial L}{\partial A_\rho} = \frac{\partial x'^\nu}{\partial x^\rho} \frac{\partial L}{\partial A_\rho}$$

odakle sledi da se $\frac{\partial L}{\partial A_\nu}$ transformiše kao kontravarijantni vektor

Za kontravarijantne drugog reda tenzore zakon transformacije je

$$\partial_\rho A_\sigma = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \partial'_\mu A'_\nu \qquad (**)$$

pa je

$$\frac{\partial \mathcal{L}}{\partial(\partial'_\mu A'_\nu)} = \frac{\partial(\partial_\rho A_\sigma)}{\partial(\partial'_\mu A'_\nu)} \frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)} = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)}$$

(*)

Odozgo je jasno da se $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}$ transformiše kao kontravarijantni tenzor

8. Pokazati da se determinanta metričkog tenzora $g_{\mu\nu}$ ne transformiše kao skalar

---

Budući da je коваријантни $g_{\mu\nu}$ tenzor, transformiše se na sledeći način

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} .$$

Determinanta metričkog tenzora se transformiše na sledeći način

$$\bar{g} = \det(g'_{\mu\nu}) = \det(g_{\alpha\beta}) \det\left(\frac{\partial x^\alpha}{\partial x'^\mu}\right) \cdot \det\left(\frac{\partial x^\beta}{\partial x'^\nu}\right) = g\left[\det\left(\frac{\partial x^\alpha}{\partial x'^\mu}\right)\right]^2$$

Dakle, nakon transformacije nije ispunjeno

$$\bar{g} = g$$

$g = \det(g_{\alpha\beta})$ se ne transformiše kao skalar, односно није Лоренцов скалар

1a. Polazeći od Lorencovih transformacija za kontravarijantne vektore (u pravcu x ose) i koristeći pravilo

$$\frac{\partial A}{\partial u} = \frac{\partial \phi}{\partial u}\frac{\partial A}{\partial \phi} + \frac{\partial \psi}{\partial u}\frac{\partial A}{\partial \psi}, \quad (*)$$

$A = A[\phi(u,v), \psi(u,v)]$, pokazati da je

$$\frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)$$

$$\frac{\partial}{\partial t} = \gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)$$

Potom izvesti izraz za y-komponentu $\text{rot}\,\vec{A}$ izražene preko operatora $\frac{\partial}{\partial x'}$ i $\frac{\partial}{\partial t'}$.

Iz $(*)$ i pridruživanja

$$\phi = x', \ \psi = t', \quad u = x, \ v = t$$

sledi

$$\frac{\partial A}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial A}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial A}{\partial t'}$$

Lorencove transformacije (boost-ovi) u pravcu x ose glase

$$x' = \gamma(x - vt) \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right) \quad \left[\text{ili} \quad \frac{\partial}{\partial x^1} = \gamma\left(\frac{\partial}{\partial x^{1'}} - \beta\frac{\partial}{\partial x^{0'}}\right)\right]$$

Slično,

$$\frac{\partial}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -\gamma v\frac{\partial}{\partial x'} + \gamma\frac{\partial}{\partial t'}$$

$$= \gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)$$

$$\left[\text{ili} \quad \gamma c\left(\frac{\partial}{\partial x^{0'}} - \beta\frac{\partial}{\partial x^{1'}}\right)\right]$$

$$\text{rot}\,\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\left(\text{rot}\,\vec{A}\right)_y = -\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)$$

Budući da je $y = y'$ i $z = z'$, te $\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$ jasno je da je

$$\left(\text{rot}\,\vec{A}\right)_y = -\gamma\left[\frac{\partial A_z}{\partial x'} - \cancelto{0}{\frac{v}{c^2}\frac{\partial A_z}{\partial t'}}\right] + \frac{\partial A_x}{\partial z'}$$

$$= \frac{\partial A_x}{\partial z'} - \gamma\frac{\partial A_z}{\partial x'}$$

9 Pokazati da se Lorencove transformacije (bezostovi) mogu parametrizovati hiperboličkim f-jama. Argument hiperboličke f-je povezati sa relativnom brzinom referentnih sistema S i S'. Pretpostaviti da je transformacija linearna i koristiti definiciju skalarnog proizvoda u prostoru Minkovskog.

$$S \quad (x^0, x^1) \quad \xrightarrow{\boxed{x' = A x}} \quad S' \quad (x'^0, x'^1)$$

$$x^0 = a x'^0 + b x'^1 \qquad (1)$$
$$x^1 = c x'^0 + d x'^1$$

Euklidski prostor $\langle x, y \rangle = \delta_{ij} \xi_1^i \xi_2^j$

po analogiji

Pseudo-euklidski prostor $\langle x, y \rangle = g_{\alpha\beta} \xi_1^\alpha \xi_2^\beta$

(prostor Minkovskog) (2)

Odnosno

$$\langle x, x \rangle = g_{\alpha\beta} \xi_1^\alpha \xi_1^\beta$$ — skalar, Lorenc invarijanta

$$x^T g x = \text{const} = x'^T g x' = x^T A^T g A x \Rightarrow$$

$$g = A^T g A \qquad (3)$$

Iz (3) $\Rightarrow$ $\det A = \pm 1$ (izometrija)

$\Updownarrow$

npr. Rotacije iz 3D

Iz (1) sledi da je

Lambda

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

a $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ pa je

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a & -c \\ b & -d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a^2 - c^2 & ab - cd \\ ba - cd & b^2 - d^2 \end{pmatrix}$$

Odozgo sledi sistem j-na

$a^2 - c^2 = 1$ $\qquad a \neq 0$

$ab - cd = 0$

$b^2 - d^2 = -1$

Ako stavimo $\boxed{\beta = \frac{c}{a}}$

$$a^2 - c^2 = 1$$

$$1 - \beta^2 = \frac{1}{a^2} \Rightarrow a^2 = \frac{1}{1-\beta^2} \Rightarrow \boxed{a = \pm \frac{1}{\sqrt{1-\beta^2}}}$$

$$ab = cd$$

$$\frac{b}{d} = \frac{c}{a} = \beta \Rightarrow \boxed{b = \beta d}$$

$$b^2 - d^2 = -1$$

$$\beta^2 - 1 = -\frac{1}{d^2} \Rightarrow \frac{1}{d^2} = 1 - \beta^2 \Rightarrow \boxed{d = \pm \frac{1}{\sqrt{1-\beta^2}}}$$

Dakle,

NE

$$\Lambda = \pm \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & \frac{\beta}{\sqrt{1-\beta^2}} \\ \frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix} \equiv \begin{bmatrix} \Lambda^0_{\ 0} & \Lambda^0_{\ 1} \\ \Lambda^1_{\ 0} & \Lambda^1_{\ 1} \end{bmatrix} \quad (4)$$

NE

Od interesa su Λ-ovi za koje je $\det \Lambda = 1$ i $\Lambda^0_{\ 0} \geq 1$ (to su transformacije koje se mogu dobiti infinitezimalnim promenama iz 1-ce, tj. identične matrice)

Ako se uvede da je $\beta = th\,\varphi$, odgovarajuća matrica se može zapisati kao

$$A = \begin{pmatrix} ch\,\varphi & sh\,\varphi \\ sh\,\varphi & ch\,\varphi \end{pmatrix}$$

Radi jednostavnosti, zadatak je rađen samo za transformaciju x^0 i x^1 komponenti kvadrivektora. Ako se uzmu u obzir i ostale komponente, L će glasiti:

$$L = \begin{pmatrix} ch\,\varphi & sh\,\varphi & 0 & 0 \\ sh\,\varphi & ch\,\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Za domaći:

Pokazati

$$ch\,\varphi = \frac{1}{\sqrt{1 - th^2\varphi}}$$

$$\boxed{ch^2\varphi - sh^2\varphi = 1}$$

$$sh\,\varphi = \frac{th\,\varphi}{\sqrt{1 - th^2\varphi}}$$

Klasična teorija polja

Lagranžev formalizam

$\mathcal{L}$ – gustina lagranžijana (skalar)

$L = \int \mathcal{L} \, d^3\vec{x}$; tipično $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$

Ojler-Lagranževa j-na za polje $\varphi(x)$

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0$$

Za dato $\varphi(x)$, konjugovani impuls (konjugovano polje) je

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \leftarrow \text{gustina}$$

Znajući $\pi(x)$, gustina hamiltonijana je

$$\mathcal{H} = \pi(x)\dot{\varphi}(x) - \mathcal{L}$$

i važi

$$H = \int \mathcal{H} \, d^3\vec{x}$$

1. Naći j-ne kretanja i Hamiltonijan* koji odgovaraju Lagranžijanu**

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu \varphi)^2 - m^2\varphi^2\right)$$

*, ** } misli se na gustinE

Oiler-Lagranževa j-na kretanja

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right) = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = ?$$

$$(\partial_\mu \varphi)^2 = (\partial_\mu \varphi)(\partial^\mu \varphi) = (\partial_\mu \varphi) g^{\mu\nu} (\partial_\nu \varphi)$$

Iz datog Lagranžijana

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{\partial}{\partial(\partial_\mu \varphi)}\left(\frac{1}{2}\left\{(\partial_\mu \varphi)^2 - m^2\varphi^2\right\}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial(\partial_\mu \varphi)}(\partial_\mu \varphi) g^{\mu\nu}(\partial_\nu \varphi) - \underbrace{\cancel{\frac{1}{2}\frac{\partial}{\partial(\partial_\mu \varphi)}(m^2\varphi^2)}}_{0}$$

$$= \frac{1}{2}\left(g^{\mu\nu}(\partial_\nu \varphi) + g^{\mu\nu}(\partial_\nu \varphi)\right) = \partial^\mu \varphi \qquad (2)$$

$$\frac{\partial L}{\partial \varphi} = -m^2 \varphi \qquad (3)$$

Iz (1), (2) i (3)

$$0 = -m^2 \varphi - \partial_\mu (\partial^\mu \varphi)$$, odnosno

$$\partial_\mu (\partial^\mu \varphi) + m^2 \varphi = 0$$ [J-na kretanja]

Gustina Hamiltonijana

$$\mathcal{H} = \pi(x)\dot{\varphi}(x) - \mathcal{L}$$

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$ Gustina impulsa

$$\pi(x) = \frac{\partial L}{\partial(\partial_0 \varphi)} = \quad \partial_0 \varphi = \dot{\varphi} \checkmark$$

$\longrightarrow$ (isto što i ∂^0)

$$\mathcal{H} = \dot{\varphi}^2 - \frac{1}{2}\left((\partial_\mu \varphi)^2 - m^2\varphi^2\right)$$

$$= (\partial_0 \varphi)^2 - \frac{1}{2}\left((\partial_\mu \varphi)^2 - m^2\varphi^2\right)$$

$\partial_\mu \partial^\mu = \Box \rightarrow$ Dalamberov operator

J-na kretanja uz pomoć $\Box$

$$\Box \varphi + m^2 \varphi = 0$$

2. Izvesti sinusnu Gordonovu j-nu iz lagranžija na

$$\mathcal{L} = \frac{1}{2}\left((\partial_t\varphi)^2 - (\partial_x\varphi)^2\right) + \cos\varphi$$

U pitanju je 1D problem. Uzeti da je $c = 1$.

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = 0 \qquad (1)$$

$$\frac{\partial\mathcal{L}}{\partial\varphi} = -\sin\varphi \qquad (2)$$

U ovom slučaju

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = \partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)}\right) - \partial_x\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\varphi)}\right) \qquad (3)$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)} = \frac{\partial}{\partial(\partial_t\varphi)}\left[\frac{1}{2}\left((\partial_t\varphi)^2 - (\partial_x\varphi)^2\right) + \cos\varphi\right] \qquad (4)$$

$$= \partial_t\varphi$$

i slično

$$\frac{\partial\mathcal{L}}{\partial(\partial_x\varphi)} = \frac{\partial}{\partial(\partial_x\varphi)}\left[\frac{1}{2}\left((\partial_t\varphi)^2 - (\partial_x\varphi)^2\right) + \cos\varphi\right] \qquad (5)$$

$$= -\partial_x\varphi \left(= \partial^x\varphi\right)$$

uporediti sa prvim zadatkom iz oblasti

Koristeći (3), (4) i (5)

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \varphi)} \right) = \partial_t(\partial_t \varphi) - \partial_x(\partial_x \varphi)$$

$$= \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = \Box \varphi$$

Dakle, jednačina kretanja glasi

$$\Box \varphi + \sin \varphi = (\Box + \sin)\varphi = 0$$

3. Dat je Lagranžijan koji opisuje dva realna skalarna polja $\phi_1(X)$ i $\phi_2(X)$

$$\mathcal{L} = \frac{1}{2}\left[(\partial\phi_1)^2 + (\partial\phi_2)^2\right] - \frac{m^2}{2}(\phi_1^2+\phi_2^2) - \frac{\lambda}{4!}(\phi_1^2+\phi_2^2)^2.$$

Izvesti odgovarajuće jednačine kretanja.

Ojler - Lagranžova j-na kretanja, za neko φ

$$\frac{\partial L}{\partial \varphi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \varphi)}\right) = 0$$

$$\frac{\partial L}{\partial(\partial_\mu\phi_1)} = \frac{1}{2}\frac{\partial}{\partial(\partial_\mu\phi_1)}\left[(\partial_\nu\varphi_1)(\partial_\rho\varphi_1)g^{\nu\rho}\right]$$

$$= \frac{1}{2}\left(\delta^\mu_\nu\,\partial_\rho\phi_1\, g^{\nu\rho} + \partial_\nu\varphi_1\,\delta^\mu_\rho\, g^{\nu\rho}\right)$$

$$= \frac{1}{2}\left(\partial^\mu\phi_1 + \partial^\mu\phi_1\right) = \partial^\mu\phi_1$$

Slično se dobija i za drugo polje

$$\frac{\partial L}{\partial(\partial_\mu\phi_2)} = \dots = \partial^\mu\phi_2$$

$$\frac{\partial L}{\partial \phi_1} = -m^2\phi_1 - \lambda\phi_1^3 + \lambda\varphi_1\varphi_2^2$$

$$= -m^2\phi_1 - \frac{\lambda}{6}\left(\phi_1^3 + \varphi_1\varphi_2^2\right)$$

$$\frac{\partial L}{\partial \phi_2} = \cdots = -m^2\phi_2 - \lambda\left(\phi_2^3 + \phi_1^2\phi_2\right)$$? PROVERITI

Једначине кретања гласе:

$$\Box\phi_1 = -m^2\phi_1 - \frac{\lambda}{6}\left(\phi_2^3 + \phi_1\phi_2^2\right)$$

$$\Box\phi_2 = -m^2\phi_2 - \frac{\lambda}{6}\left(\phi_2^3 + \phi_1^2\phi_2\right)$$